On the relation between classical and quantum critical systems

1993 J. Phys. A: Math. Gen. 26 L749

(http://iopscience.iop.org/0305-4470/26/16/007)

J. Phys. A: Math. Gen. 26 (1993) L749–L752. Printed in the UK

LETTER TO THE EDITOR

On the relation between classical and quantum critical systems

Carlos H C Moreira and Ricardo S Schor

Departamento de Física, Instituto de Ciências Exatas, Universidade Federal de Minas Gerais, CP 702, CEP 30161-970 Belo Horizonte, Brazil

Received 3 June 1993

Abstract. We consider critical asymptotically hierarchical quantum models and show that the probability distribution of the appropriately scaled square of the total spin converges, as the number of spins tends to infinity, to the same function in the corresponding classical systems.

It is generally believed, mainly based on heuristic arguments [5], that the critical behaviour of a statistical mechanical system does not depend on the way, classical or quantum-mechanical, employed to describe it. As a consequence, one should expect quantum-mechanical models to fall into the same universality classes as their classical correspondents, presenting the same critical exponents and scaling functions. Sewell [7] has made use of the fact that the components of the L-block spin variables critically scaled (non-central limit scaling) commute in the limit $L \to \infty$ to show quite generally that the long-distance behaviour of critical quantum systems is classical in the sense that it can be formulated in terms of probability measures. However, the question of the relation between this behaviour and the one corresponding to the classical version of the system still lacks rigorous proof.

Here we consider the spin-$\frac{1}{2}$ case of a certain class of models, called asymptotically hierarchical models in the terminology of [8], and take as the classical analogue of these models those obtained by replacing each quantum-mechanical spin by a classical three-dimensional vector-valued variable in the Hamiltonian. Specifically, we study the probability distribution of the appropriately scaled square of the total spin for critical quantum systems and show that it converges, as the number of spins tends to infinity, to the same function in the corresponding classical systems. Thus we exibit explicitly a property of statistical-mechanical systems which at the critical point does not depend on whether one uses a classical or quantum-mechanical description.

The asymptotically hierarchical models are generalizations of Dyson's hierarchical models, originally introduced in [4]. The classical version of these models, with spins taking values ± 1, at the critical point was first studied in a rigorous way in the fundamental paper by Bleher and Sinai [3]. Vector-valued classical spins at low temperatures were analysed by Bleher and Major in [1, 2] and by Schor and O'Carroll in [6], using a different hierarchical model.

To each site of the lattice $\Lambda_n = \{1, 2, \ldots, 2^n\}$ associate a spin variable which may be a classical or quantum-mechanical variable. A member of the class of asymptotically hierarchical models is specified by an initial Hamiltonian $H_{n_0}(S_1, \ldots, S_{2^{n_0}})$ defined on Λ_{n_0}.

The Hamiltonians $H_n(S_1, \ldots, S_{2^n})$ on Λ_n for $n > n_0$ are given recursively by

$$H_n(S_1, \ldots, S_{2^n}) = H_{n-1}(S_1, \ldots, S_{2^{n-1}}) + H_{n-1}(S_{2^{n-1}+1}, \ldots, S_{2^n}) - \left(\frac{c}{4}\right)^n \left(\sum_{i\in\Lambda_n} S_i\right)^2. \tag{1}$$

We will assume that H_{n_0} is rotationally invariant. As shown in Dyson's paper [4], the thermodynamic limit exists if $c < 2$ and there is a phase transition if $c > 1$. We assume in the following $1 < c < 2$.

Let $p_n^{(\mathrm{cl})}(.; \beta)$ be the distribution of $\sum_{i\in\Lambda_n} S_i$ (which depends only on $|\sum_{i\in\Lambda_n} S_i|$) in the classical case at inverse temperature β. It follows directly from (1) that it satisfies the recursion

$$p_n^{(\mathrm{cl})}(s; \beta) = \frac{Z_{n-1}^{(\mathrm{cl})}(\beta)^2}{Z_n^{(\mathrm{cl})}(\beta)} \exp\left[\beta \left(\frac{c}{4}\right)^n s^2\right] \int p_{n-1}^{(\mathrm{cl})}(|\boldsymbol{u}|; \beta)\, p_{n-1}^{(\mathrm{cl})}(|s\hat{e}_1 - \boldsymbol{u}|; \beta)\, \mathrm{d}^3 u \tag{2}$$

where $\hat{e}_1$ is the unit vector in the one-direction and $Z_m^{(\mathrm{cl})}(\beta)$ is the classical canonical partition function on the Λ_m lattice.

Proceeding heuristically as in [8], we expect that above the critical temperature $|s|^2 \sim 2^n$ as $n \to \infty$ and hence the exponential in (2) should be irrelevant in this case. On the other hand, below the critical temperature $|\boldsymbol{s}|^2 \sim 2^{2n}$ and the exponential is dominant. At the critical temperature we may expect $|s|^2 \sim (4/c)^n$. Let $\Delta_n = (\sqrt{c}/2)^n$ and define

$$g_n^{(\mathrm{cl})}(x; \beta) = \frac{8\pi}{c} \Delta_n^{-3} Z_n^{(\mathrm{cl})}(\beta) x p_n^{(\mathrm{cl})}(\Delta_n^{-1} x; \beta). \tag{3}$$

In terms of the quantity above, we may express the canonical ensemble average of a function of the total spin properly scaled as

$$\left\langle F\left(\Delta_n \sum_{i\in\Lambda_n} S_i\right)\right\rangle_n^{(\mathrm{cl})} = \frac{\int_0^\infty F(x) x g_n^{(\mathrm{cl})}(x; \beta)\, \mathrm{d}x}{\int_0^\infty x g_n^{(\mathrm{cl})}(x; \beta)\, \mathrm{d}x}. \tag{4}$$

From (2) the following recurrence may be inferred:

$$g_n^{(\mathrm{cl})}(x; \beta) = \frac{1}{\pi\sqrt{c}} x \exp(\beta x^2) \int \frac{g_{n-1}^{(\mathrm{cl})}(|\boldsymbol{r}|; \beta)}{|\boldsymbol{r}|} \frac{g_{n-1}^{(\mathrm{cl})}(|(2/\sqrt{c}) x\hat{e}_1 - \boldsymbol{r}|; \beta)}{|(2/\sqrt{c}) x\hat{e}_1 - \boldsymbol{r}|}\, \mathrm{d}^3 r. \tag{5}$$

Defining the new variables $v_1 = r$ and $v_2 = \sqrt{(4/c)x^2 + r^2 - (4/\sqrt{c}) x r \cos\theta}$, one may write

$$g_n^{(\mathrm{cl})}(x; \beta) = \exp(\beta x^2) \iint_{|v_1 - v_2| \leqslant (2/\sqrt{c})x \leqslant v_1 + v_2} g_{n-1}^{(\mathrm{cl})}(v_1; \beta) g_{n-1}^{(\mathrm{cl})}(v_2; \beta)\, \mathrm{d}v_1\, \mathrm{d}v_2. \tag{6}$$

The Gaussian fixed point of the recursion above is

$$g(x; \beta) = \sqrt{\frac{8c}{\pi}} \alpha(\beta)^{3/2} x \exp[-\alpha(\beta) x^2] \tag{7}$$

where $\alpha(\beta) = \beta c/(2-c)$.

Using the techniques of [8] one may show that the convergence of $g_n^{(cl)}(.;\beta)$ to the distribution above really occurs in the sense of probabilities for $\sqrt{2} < c < 2$. The reason for this interval may be found in the remark to the theorem stated below.

We consider now the quantum case. Let $\Lambda_p^{(j)} = \{(j-1)2^p+1, (j-1)2^p+2, \ldots, (j-1)2^p+2^p\}$ for $1 \leqslant p \leqslant n$ and $1 \leqslant j \leqslant 2^{n-p}$. Define

$$\mu_p^{(j)} = \sum_{i\in\Lambda_p^{(j)}} S_i. \tag{8}$$

Then, as pointed out in Dyson's original paper [4], the set

$$\{(\mu_p^{(j)})^2 : 1 \leqslant p \leqslant n, 1 \leqslant j \leqslant 2^{n-p}; \mu_n^{(1),3}\}$$

(where $\mu_n^{(1),3}$ is the third component of $\mu_n^{(1)}$) is maximally commuting in $\mathbf{C}^{2^n}$ and the common eigenfunctions diagonalize H_n if H_{n_0} depends only on $\mu_p^{(j)}$, $1 \leqslant p \leqslant n_0$, $1 \leqslant j \leqslant 2^{n_0-p}$. Denoting the eigenfunctions by $|\{\ell_p^{(j)}\}; m\rangle$, we have

$$\begin{aligned}(\mu_r^{(k)})^2|\{\ell_p^{(j)}\}; m\rangle &= \ell_r^{(k)}(\ell_r^{(k)}+1)|\{\ell_p^{(j)}\}; m\rangle \\ \mu_n^{(1),3}|\{\ell_p^{(j)}\}; m\rangle &= m|\{\ell_p^{(j)}\}; m\rangle.\end{aligned} \tag{9}$$

Since $\mu_p^{(j)} = \mu_{p-1}^{(2j-1)} + \mu_{p-1}^{(2j)}$ $(1 \leqslant p \leqslant n)$, we must have $|\ell_{p-1}^{(2j-1)} - \ell_{p-1}^{(2j)}| \leqslant \ell_p^{(j)} \leqslant \ell_{p-1}^{(2j-1)} + \ell_{p-1}^{(2j)}$ and initially $\ell_1^{(j)} \in \{0, 1\}$. Also, the possible values of m are restricted to $|m| \leqslant \ell_n^{(1)}$.

In terms of the basis (9) the quantum canonical ensemble average of a function of $(\mu_n^{(1)})^2 = (\sum_{i\in\Lambda_n} S_i)^2$ is

$$\langle F((\mu_n^{(1)})^2)\rangle_n^{(q)} = \sum_{\ell=0}^{2^{n-1}} F(\ell(\ell+1))(2\ell+1)p_n^{(q)}(\ell;\beta) \tag{10}$$

where $p_n^{(q)}(.;\beta) : \{0, \ldots, 2^{n-1}\} \longrightarrow \mathbf{R}$ is the distribution of $\ell_n^{(1)}$. Again, from (1) we get the recursion

$$p_n^{(q)}(\ell;\beta) = \frac{Z_{n-1}^{(q)}(\beta)^2}{Z_n^{(q)}(\beta)} \exp\left[\beta\left(\frac{c}{4}\right)^n \ell(\ell+1)\right] \sum_{\substack{\ell_i\in\{0,\ldots,2^{n-2}\} \\ |\ell_1-\ell_2|\leqslant\ell\leqslant\ell_1+\ell_2}} p_{n-1}^{(q)}(\ell_1;\beta)p_{n-1}^{(q)}(\ell_2;\beta). \tag{11}$$

Let

$$g_n^{(q)}(x;\beta) = (4/c)\Delta_n^{-2} Z_n^{(q)}(\beta) p_n^{(q)}(\Delta_n^{-1}x;\beta) \tag{12}$$

for $x \in A_n$, where $A_n = \left\{\Delta_n\ell : 0 \leqslant \ell \leqslant 2^{n-1}\right\}$. The function defined above is analogous to $g_n^{(cl)}(.;\beta)$ in the classical case, as may be seen by expressing ensemble averages as

$$\langle F((\Delta_n\mu_n^{(1)})^2)\rangle_n^{(q)} = \frac{\Delta_n \sum_{x\in A_n} F(x(x+\Delta_n))(x+\frac{1}{2}\Delta_n)g_n^{(q)}(x;\beta)}{\Delta_n \sum_{x\in A_n}(x+\frac{1}{2}\Delta_n)g_n^{(q)}(x;\beta)} \tag{13}$$

and comparing with (4). One may also check whether the recursion

$$g_n^{(q)}(x;\beta) = \exp\left[\beta x(x+\Delta_n)\right]\Delta_{n-1}^2 \sum_{\substack{x_1,x_2\in A_{n-1} \\ |x_1-x_2|\leqslant\frac{2}{\sqrt{c}}x\leqslant x_1+x_2}} g_{n-1}^{(q)}(x_1;\beta)g_{n-1}^{(q)}(x_2;\beta) \tag{14}$$

is formally identical to the classical one in the limit $n \to \infty$. Thus, provided the heuristic arguments for the quantum case can be justified, we see that the distribution of $\lim_{n\to\infty}(\Delta_n \sum_{i\in\Lambda_n} S_i)^2$ at the critical temperature is the same whether one uses classical or quantum statistical mechanics.

Theorem. For each $2^{1/2} < c < 2^{4/5}$ there is a non-void set of initial Hamiltonians such that $g_n^{(q)}(x; \beta_{cr})$ (appropriately normalized) converges in probability to $\sqrt{8c\pi^{-1}}\,\alpha(\beta_{cr})^{3/2} x \exp[-\alpha(\beta_{cr})x^2]$ at critical inverse temperature $\beta_{cr} > 0$.

Remark. The study of the flux defined by transformation (6) in the vicinity of the Gaussian fixed point shows that there is one and just one relevant direction for $2^{1/2} < c < 2$. Provided this relevant direction can be properly controlled, one might expect to prove the existence and thermodynamical stability of the fixed point. However, within our method of proof, which consists in studying the recursion (14) adapting the methods of the scalar $S_i = \pm 1$ case [8], the region $2^{4/5} \leqslant c < 2$ could not be handled.

Proof (sketch). We deal separately with large fields and also with very small fields, so that the perturbative region corresponds to intermediate values. Fix n_0 and suppose that $g_{n_0}^{(q)}(.; \beta)$ is close enough to the fixed point (7) in the sense that it admits a good bound on the small fields region, decays quickly enough for large fields and in the intermediate region the perturbation to the fixed point may be expanded in the basis provided by the eigenfunctions of the linearized transformation up to order N (N large but fixed) with an adequate bound on the remainder term. We prove inductively that if $g_k^{(q)}(.; \beta)$ satisfies similar conditions for $n_0 \leqslant k \leqslant n$ and $\beta \in [\beta_-^{(k)}, \beta_+^{(k)}]$, it satisfies the same conditions for $k = n+1$ and $\beta \in [\beta_-^{(n+1)}, \beta_+^{(n+1)}] \subset [\beta_-^{(n)}, \beta_+^{(n)}]$ with improved bounds. The choice of these intervals must be such as to control the growth in the relevant direction, according to the Bleher–Sinai mechanism [8]. The small-field region shrinks exponentially with n and the intermediate region grows as $\sqrt{n}$. In the intermediate region we must control the perturbative terms setting up a compromise between L_2 decay and uniform decay by means of a lemma of Tauberian type. At each step, a careful analysis of the error terms is required. These terms arise due to the linear approximation, the finiteness of the perturbative region and the approximation of sums by integrals. It turns out that the latter can only be successfully handled if $c < 2^{4/5}$, since they are genuinely of order $\Delta_n = (\sqrt{c}/2)^n$ and therefore naturally greater than the controllable part of the perturbation, which decays as $(2/c^2)^n$, for $c > 2^{4/5}$. Iteration of the induction hypothesis eventually leads to the proof of the theorem, with $\beta_{cr} \in \bigcap_{n=n_0}^{\infty} [\beta_-^{(n)}, \beta_+^{(n)}]$.

A detailed account of the proof will appear elsewhere.

This work was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil.

References

[1] Bleher P M and Major P 1984 *Commun. Math. Phys.* **95** 487
[2] Bleher P M and Major P 1988 *Ann. Inst. Henri Poincaré, Phys. Théor.* **49** 7
[3] Bleher P M and Sinai Ya G 1973 *Commun. Math. Phys.* **33** 23
[4] Dyson F 1969 *Commun. Math. Phys.* **12** 91
[5] Fisher M E 1974 *Rev. Mod. Phys.* **46** 597
[6] Schor R S and O'Carroll M 1991 *Commun. Math. Phys.* **138** 487
[7] Sewell G L 1986 *Quantum Theory of Collective Phenomena* (Oxford: Clarendon)
[8] Sinai Ya G 1982 *Theory of Phase Transitions: Rigorous Results* (London: Pergamon)